HOLLOWING THE SPHERE CALCULATIONS

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This is a supplemental document to the post, The Illusion of Growth, on the A Beautiful Aching blog, abeautifulaching.com.

INTRODUCTION

The process of Hollowing the Sphere is based on the premise of starting off with a solid sphere, hollowing it out and the material that is removed from the core is then evenly applied to the surface of the original sphere, thus increasing its outer diameter.

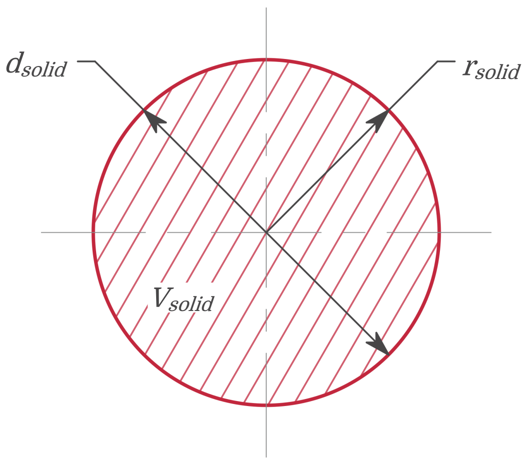
This is subsequently repeated by removing more material from within the now hollow sphere and again applying it to the surface of the newly enlarged sphere.

Exploring and developing a logical and systemic mathematical model of the process is the purpose of this document. It is intended to ascertain any underlying patterns which may be present and to support the material that is expressed in the original post, The Illusion of Growth.

The full step by step methodology will be presented deeper within this document, but first some foundations will be established.

THE BASICS

SOLID SPHERE



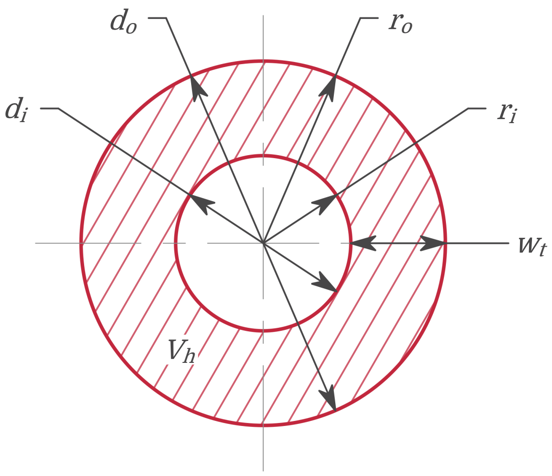
SOLID SPHERE

The equation to determine the volume of a solid sphere is,

$$V_{\text{solid}} = \frac{4\pi}{3} r_{\text{solid}}^3$$

Where, $\boldsymbol{V_{\text{solid}}}$ = Solid sphere volume.
$\boldsymbol{r_{\text{solid}}}$ = Solid sphere radius.

HOLLOW SPHERE



HOLLOW SPHERE

Determining the material volume of a hollow sphere requires that the hollow volume be subtracted from the solid volume.

The solid volume of the sphere is found by evaluating,

$$V_o = \frac{4\pi}{3} r_o^3$$

Where, $\boldsymbol{V_o}$ = Solid volume of a hollow sphere.
$\boldsymbol{r_o}$ = Outer radius of a hollow sphere.

The hollow volume of the sphere is then found by evaluating,

$$V_i = \frac{4\pi}{3} r_i^3$$

Where, $\boldsymbol{V_i}$ = Hollow volume of a hollow sphere.
$\boldsymbol{r_i}$ = Inner radius of a hollow sphere.

The material volume of the hollow sphere is then determined by subtracting the hollow volume from the solid volume.

$$V_h = V_o - V_i$$

Where, $\boldsymbol{V_h}$ = Material volume of the hollow sphere.

This process can be combined into a single equation as shown below.

$$V_h = \frac{4\pi}{3} r_o^3 - \frac{4\pi}{3} r_i^3$$

Simplifying this equation produces the following form,

$$V_h = \frac{4\pi}{3} \left(r_o^3 - r_i^3\right)$$

CONSTANT VOLUME

For this process the material volume of all the spheres remains constant. No new material is added and no material is taken away.

Therefore, the volume of the solid sphere is the same as the volume of the hollow sphere at all subsequent iterations.

Expressed in equation form gives,

$$V_{\text{solid}} = V_h$$

Replacing the volume terms with the radius base terms gives,

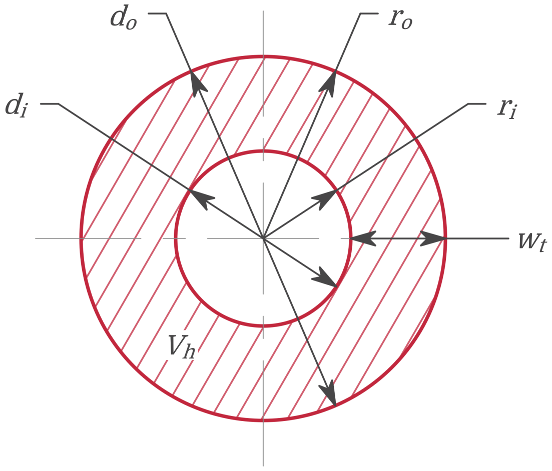
$$\frac{4\,\pi}{3}\,r_{\text{solid}}^3 = \frac{4\,\pi}{3}\left(r_o^3 - r_i^3\right)$$

Dividing both sides by $\frac{4\pi}{3}$ leads to the Constant Volume Radius equation,

$$r_{\text{solid}}^3 = r_o^3 - r_i^3$$

DETERMINING DIAMETER FROM WALL THICKNESS

A short cut method for determining the hollow sphere outer diameter based on a specified wall thickness will be shown in this section. This simplified approach was utilised to produce Table 1 of condensed steps in the blog post, The Illusion of Growth.



HOLLOW SPHERE

The wall thickness of a hollow sphere is the difference between the outer radius and the inner radius.

$$w_t = r_o - r_i$$

Where, w_t = Wall thickness of a hollow sphere.

Rearranging this in terms of the inner radius gives,

$$r_i = r_o - w_t$$

Substituting this expression of the inner radius into the Constant Volume Radius equation,

$$r_{\text{solid}}^3 = r_o^3 - r_i^3$$

results in,

$$r_{\text{solid}}^3 = r_o^3 - (r_o - w_t)^3$$

Expanding this out leads to,

$$r_{\text{solid}}^3 = r_o^3 - \left(r_o^3 - 3\, w_t\, r_o^2 + 3\, w_t^2\, r_o - w_t^3\right)$$

$$r_{\text{solid}}^3 = r_o^3 - r_o^3 + 3\, w_t\, r_o^2 - 3\, w_t^2\, r_o + w_t^3$$

$$r_{\text{solid}}^3 = 3\, w_t\, r_o^2 - 3\, w_t^2\, r_o + w_t^3$$

Subtracting r_{solid}^3 from both sides becomes,

$$3\, w_t\, r_o^2 - 3\, w_t^2\, r_o + w_t^3 - r_{\text{solid}}^3 = 0$$

This is now in the form of a quadratic equation and so the Quadratic Formula can be used to calculate a result.

For a quadratic function in the form,

$$f(x) = a\, x^2 + b\, x + c$$

Provided that $\left(b^2 - 4\, a\, c\right) \geqslant 0$, the Quadratic Formula for the zeros of this equation is,

$$x = \frac{-b \pm \sqrt{b^2 - 4\, a\, c}}{2\, a}$$

So, from the radius and wall thickness equation,

$$3\, w_t\, r_o^2 - 3\, w_t^2\, r_o + w_t^3 - r_{\text{solid}}^3 = 0$$

The parameters for the Quadratic Formula are,

$$a = 3\, w_t$$

$$b = -3\, w_t^2$$

$$c = w_t^3 - r_{\text{solid}}^3$$

From this, the Quadratic Formula with reference to the outer radius of a hollow sphere, r_o, takes the form of,

$$r_o = \frac{3\, w_t^2 \pm \sqrt{\left(-3\, w_t^2\right)^2 - 4\left(3\, w_t\right)\left(w_t^3 - r_{\text{solid}}^3\right)}}{2\left(3\, w_t\right)}$$

This then reduces down to two possible solution for the outer radius of the hollow sphere, r_o, in terms of the initial radius of the solid sphere, r_{solid} and the current wall thickness of the hollow sphere, w_t.

$$r_o = \frac{3\, w_t^2 + \sqrt{12\, w_t\, r_{\text{solid}}^3 - 3\, w_t^4}}{6\, w_t}$$

and

$$r_o = \frac{3\, w_t^2 - \sqrt{12\, w_t\, r_{\text{solid}}^3 - 3\, w_t^4}}{6\, w_t}$$

By placing values into the second equation it is found to give negative solutions for the outer radius, r_o. These negative values are physically not possible so the viable solutions come from the first equation,

$$r_o = \frac{3\ w_t^2 + \sqrt{12\ w_t\ r_{\text{solid}}^3\ - 3\ w_t^4}}{6\ w_t}$$

To test this out, the following values in millimetres will be assigned to the equation variables,

$w_t = 10$

$d_{\text{solid}} = 100$

$r_{\text{solid}} = \frac{d_{\text{solid}}}{2}$

50

The outer radius of a hollow sphere with a 10 millimetre wall thickness and an equivalent solid material diameter of 100 millimetres or radius of 50 millimetres is,

$$r_o = N\left[\frac{3\ w_t^2 + \sqrt{12\ w_t\ r_{\text{solid}}^3\ - 3\ w_t^4}}{6\ w_t}, 4\right]$$

69.49

From this the outer diameter of the hollow sphere in millimetres is,

$d_o = N[2\ r_o]$

138.97

This answer corresponds with the value presented in Table 1 of the post, The Illusion of Growth, for the hollow sphere with a 10 millimetre wall thickness.

It is also useful to perform a check to ensure that the volume of this hollow sphere is the same as that of the solid sphere.

The volume of the solid sphere in millimetres cubed is,

$$V_{\text{solid}} = N\left[\frac{4\ \pi}{3}\ r_{\text{solid}}^3, 8\right]$$

523 598.78

Next find the inner radius of the hollow sphere for the given wall thickness.

$r_i = r_o - w_t$

59.49

Following on, the inner diameter of the hollow sphere, d_i, is,

$d_i = N[2\ r_i]$

118.97

Finally, the volume of the hollow sphere is,

$$V_h = N\left[\frac{4\ \pi}{3}\left(r_o^3 - r_i^3\right), 8\right]$$

523 598.78

To summaries these results,

For the solid sphere, the parameter values are,

The diameter of the solid sphere, $d_{\text{solid}} = 100$ mm.
The volume of the solid sphere, $V_{\text{solid}} = 523\,598.78\ \text{mm}^3$.

For the hollow sphere, the parameter values are,

The new outer diameter of the hollow sphere is, $d_o = 138.97$ mm.
The new inner diameter of the hollow sphere is, $d_i = 118.97$ mm.
The wall thickness of the hollow sphere is, $w_t = 10$ mm.
The material volume of the hollow sphere is, $V_h = 523\,598.78\ \text{mm}^3$

The volume of this hollow sphere is the same as that of the solid sphere which shows that the hollow sphere fits within the progression of the defined hollowing process.

So, if the intent is to find the outer diameter of a hollow sphere which has a particular wall thickness, based on a certain sized solid sphere, then this is the approach to be taken.

THE METHODICAL APPROACH

This methodical approach can be used to achieve the same outcome as in the previous section but it requires far more steps to reach the target wall thickness. Its benefit is simply to show that a physicalised process is possible, although it is very cumbersome.

This methodical approach was utilised to produce Table 2 in the blog post, The Illusion of Growth.

A summary of the methodology is presented below.

Iteration 1.

Step 1 - Start with a Solid Sphere Diameter.
Step 2 - Determine the Solid Sphere Volume.
Step 3 - Select the Core Diameter.
Step 4 - Calculate the Core Material Volume.
Step 5 - Calculate the Increase in Diameter.
Step 6 - Calculate the Wall Thickness.
Step 7 - Check the Volume.

Iteration 2.

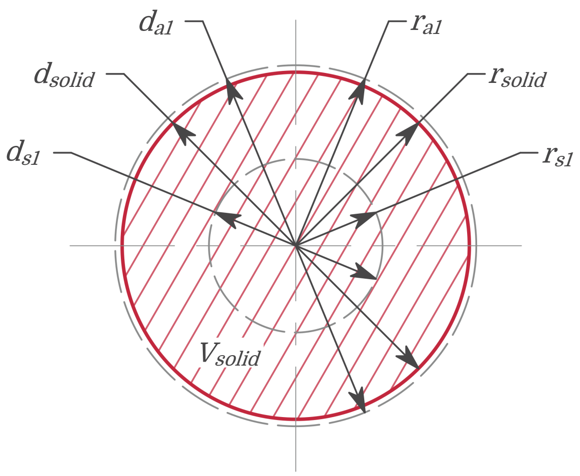
Step 1 - Determine the Diameter for Material Removal.
Step 2 - Calculate the Material Volume for Removal.
Step 3 - Calculate the Increase in Diameter.
Step 4 - Calculate the Wall Thickness.
Step 5 - Check the Volume.

Iteration 3 and on.

Repeat Iteration 2 until the desired number of iterations is reached.

The above summary will next be explained more fully.

ITERATION 1



HOLLOWING - ITERATION 1

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STEP 1 - START WITH A SOLID SPHERE DIAMETER

Start off with a solid sphere, in this case 100 millimetres in diameter.

$d_{\text{solid}} = 100$

Where, d_{solid} = Solid sphere diameter.

STEP 2 - DETERMINE THE SOLID SPHERE VOLUME

Calculate the volume of the solid sphere using the equation,

$$V_{\text{solid}} = \frac{4\,\pi}{3}\, r_{\text{solid}}^3$$

Where, V_{solid} = Solid sphere volume.
r_{solid} = Solid sphere radius.

$r_{\text{solid}} = \frac{d_{\text{solid}}}{2}$

50

$V_{\text{solid}} = N\left[\frac{4\,\pi}{3}\, r_{\text{solid}}^3, 8\right]$

523 598.78

STEP 3 - SELECT THE CORE DIAMETER

Generally, to determine the amount of core material to remove, the mean diameter of the wall thickness is chosen. In this initial case however, this is not possible as the sphere is still solid, so an equivalent value of half the sphere diameter is selected instead.

$$d_{s1} = \frac{d_{solid}}{2}$$

50

STEP 4 - CALCULATE THE CORE MATERIAL VOLUME

The volume of material to remove from the core at the chosen diameter is found by evaluating,

$$V_{s1} = \frac{4\pi}{3} r_{s1}^3$$

Where, V_{s1} = Inner material volume to be removed.
r_{s1} = Inner radius of material to be removed.

$$r_{s1} = \frac{d_{s1}}{2}$$

25

$$V_{s1} = N\left[\frac{4\pi}{3} r_{s1}^3, 7\right]$$

65 449.85

STEP 5 - CALCULATE THE INCREASE IN DIAMETER

From this core material volume, calculate the increase in the sphere diameter when that material is placed evenly on the surface of original sphere.

Essentially, there is a shell added to the surface of the sphere which also happens to be a hollow sphere. The thickness of this shell is the increase in the radial thickness of the hollow sphere.

This gives rise to the following variables to be determined,

$V_{a1} = V_{s1}$
r_{a1} = To be determined.

Where, V_{a1} = Material volume to be added to the surface of the sphere.
Which is equal to the volume removed from the core.
r_{a1} = Outer radius of the sphere with the new added material.

The equation for the addition of this shell material is,

$$V_{a1} = \frac{4\pi}{3}\left(r_{a1}^3 - r_{solid}^3\right)$$

Rearranging this to find the outer shell radius gives,

$$\frac{3\ V_{a1}}{4\pi} = r_{a1}^3 - r_{solid}^3$$

$$r_{a1}^3 = \frac{3\ V_{a1}}{4\pi} + r_{solid}^3$$

$$r_{a1} = \sqrt[3]{\frac{3\ V_{a1}}{4\pi} + r_{solid}^3}$$

Evaluating this equation with the know variables comes to,

$V_{a1} = V_{s1}$

65 449.85

$$r_{a1} = N\left[\sqrt[3]{\frac{3\ V_{a1}}{4\,\pi} + r_{solid}^{3}}\ , 6\right]$$

52.0021

Then the diameter of the now enlarged sphere is,

$d_{a1} = 2\ r_{a1}$

104.004

Where, d_{a1} = Outer diameter of the enlarged sphere.

STEP 6 - CALCULATE THE WALL THICKNESS

Calculate the wall thickness of this now hollow sphere.

$$w_{t1} = \frac{(d_{a1} - d_{s1})}{2}$$

27.0021

Where, w_{t1} = Wall thickness of the hollow sphere after Iteration 1.

STEP 7 - CHECK THE VOLUME

As a check, the material volume can be calculated for this hollow sphere to ensure that it is the same as the original solid sphere.

$$V_{h1} = N\left[\frac{4\,\pi}{3}\left(r_{a1}^{3} - r_{s1}^{3}\right), 8\right]$$

523 598.78

Where, V_{h1} = Material volume of the hollow sphere after Iteration 1.

As is evident from the result of this equation, $V_{h1} = V_{solid}$.

ITERATION 1 SUMMARY

This now leaves a hollow sphere with a wall thickness and an outer diameter that is greater than the original.

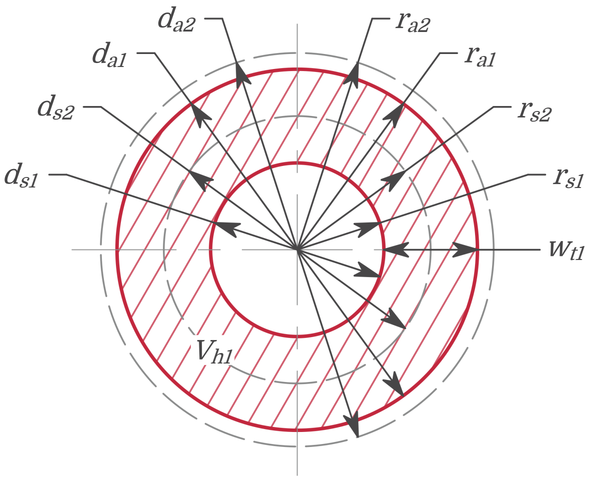
For the original solid sphere, the parameter values are,

The diameter of the solid sphere, d_{solid} = 100 mm.
The volume of the solid sphere, V_{solid} = 523 598.78 mm^3.

For the new hollow sphere, the parameter values are,

The new outer diameter of the hollow sphere is, d_{a1} = 104.00 mm.
The new inner diameter of the hollow sphere is, d_{s1} = 50 mm.
The wall thickness of the hollow sphere is, w_{t1} = 27.00 mm.
The material volume of the hollow sphere is, V_{h1} = 523 598.78 mm^3

ITERATION 2



HOLLOWING - ITERATION 2

STEP 1 - DETERMINE THE DIAMETER FOR MATERIAL REMOVAL

For the next part of the inner material to be removed, determine the mean diameter of the wall thickness by evaluating,

$$d_{s2} = d_{s1} + \frac{d_{a1} - d_{s1}}{2}$$

Where, d_{s2} = Mean diameter of the wall thickness at Iteration 2.
d_{a1} = Outer hollow sphere diameter from Iteration 1.
d_{s1} = Inner hollow sphere diameter from Iteration 1.

$$d_{s2} = d_{s1} + \frac{d_{a1} - d_{s1}}{2}$$

77.0021

STEP 2 - CALCULATE THE MATERIAL VOLUME FOR REMOVAL

Calculate the material volume of this inner portion of the sphere at the mean wall thickness diameter.

The mean wall thickness radius is required for this, so,

$$r_{s2} = \frac{d_{s2}}{2}$$

38.5010

Where, r_{s2} = Mean radius of the wall thickness at Iteration 2.

The volume of material to be removed is,

$$V_{s2} = \frac{4\,\pi}{3}\left(r_{s2}^{3} - r_{s1}^{3}\right)$$

173 610.

STEP 3 - CALCULATE THE INCREASE IN DIAMETER

From this established volume, calculate the diameter of the sphere when this material is added to the surface of the enlarged sphere.

$$r_{a2} = \sqrt[3]{\frac{3\ V_{s2}}{4\ \pi} + r_{a1}^3}$$

56.6779

$$d_{a2} = 2\ r_{a2}$$

113.356

STEP 4 - CALCULATE THE WALL THICKNESS

Calculate the wall thickness of this new hollow sphere.

$$w_{t2} = \frac{(d_{a2} - d_{s2})}{2}$$

18.1769

Where, w_{t2} = Wall thickness of the new hollow sphere after Iteration 2.

STEP 5 - CHECK THE VOLUME

As a check, the material volume can be calculated for this hollow sphere to see if it the same as the original solid sphere.

$$V_{h2} = \frac{4\ \pi}{3}\left(r_{a2}^3 - r_{s2}^3\right)$$

523 598.78

Where, V_{h2} = Material volume of the hollow sphere after Iteration 2.

As is evident from the result of this equation, $V_{h2} = V_{solid}$.

ITERATION 2 SUMMARY

This again creates a larger sphere with a larger hollow core and thinner wall thickness than previous.

For the previous hollow sphere, the parameter values are,

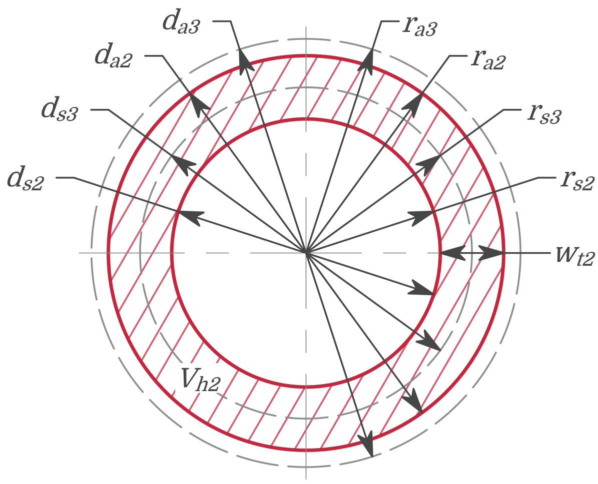
The outer diameter of the hollow sphere is, d_{a1} = 104.00 mm.
The inner diameter of the hollow sphere is, d_{s1} = 50 mm.
The wall thickness of the hollow sphere is, w_{t1} = 27.00 mm.
The material volume of the hollow sphere is, V_{h1} = 523 598.78 mm^3

For the new hollow sphere, the parameter values are,

The new outer diameter of the hollow sphere is, d_{a2} = 113.36 mm.
The new inner diameter of the hollow sphere is, d_{s2} = 77.00 mm.
The wall thickness of the hollow sphere is, w_{t2} = 18.18 mm.
The material volume of the hollow sphere is, V_{h2} = 523 598.78 mm^3.

ITERATION 3 AND ON

Repeat all the steps in Iteration 2 until the desired number of iterations is reached.



HOLLOWING - ITERATION 3

Repeating these steps continuously will generate larger and larger hollow spheres with thinner and thinner wall thicknesses.

CONCLUSION

The process of Hollowing the Sphere is shown to generate hollow spheres of ever increasing diameter and thinner wall thicknesses, while maintaining the same volume of material as the solid sphere. Which implies that it retains the same mass as the original, if a material density were to be added to the sphere.

This means that there is an increase in the overall size of the sphere but no increase in the actual substance of the sphere.

So, can the tag of The Illusion of Growth be given to an object that grows in size but gains no substantive mass?

It seems fitting.